

BOUNDS FOR EIGENVALUE RATIOS OF THE LAPLACIAN*

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ABSTRACT. For a bounded domain Ω with a piecewise smooth boundary in an n -dimensional Euclidean space \mathbf{R}^n , we study eigenvalues of the Dirichlet eigenvalue problem of the Laplacian. First we give a general inequality for eigenvalues of the Laplacian. As an application, we study lower order eigenvalues of the Laplacian and derive the ratios of lower order eigenvalues of the Laplacian.

1. INTRODUCTION

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with a piecewise smooth boundary $\partial\Omega$ in an n -dimensional Euclidean space \mathbf{R}^n . We consider the following Dirichlet eigenvalue problem of the Laplacian:

$$(1.1) \quad \begin{cases} \Delta u = -\lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It is well known that the spectrum of this problem is real and discrete:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \nearrow \infty,$$

where each λ_i has finite multiplicity which is repeated according to its multiplicity.

The investigation of universal bounds for eigenvalues of the eigenvalue problem (1.1) was initiated by Payne, Pólya and Weinberger [11]. In 1956, they showed that for $\Omega \subset \mathbf{R}^2$, the ratio of the first two eigenvalues satisfies

$$(1.2) \quad \frac{\lambda_2}{\lambda_1} \leq 3;$$

they further conjectured that one could replace the value 3 here by the value that $\frac{\lambda_2}{\lambda_1}$ assumes when Ω is a disk, approximately 2.539. With respect to the conjecture of Payne, Pólya and Weinberger, many mathematicians studied it. For examples, Brands [5], de Vries [8], Chiti [7], Hile and Protter [9], Marcellini [10] and so on. Finally Ashbaugh and Benguria [2] (cf. [1] and [3]) solved this conjecture.

For $\Omega \subset \mathbf{R}^2$, Payne, Pólya and Weinberger [11] also showed that

$$(1.3) \quad \frac{\lambda_2 + \lambda_3}{\lambda_1} \leq 6.$$

Subsequent to the paper of Payne, Pólya and Weinberger, many mathematicians improved the constant 6 in (1.3). Specifically, in 1964, Brands [5] obtained $3 + \sqrt{7}$; then in 1980, Hile and Protter [9] obtained 5.622; Marcellini [10] obtained $\frac{15+\sqrt{345}}{6}$; and very recently, Chen and Zheng

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[6] have obtained 5.3507. Furthermore, under the condition $\frac{\lambda_2}{\lambda_1} \geq 2 - \frac{\lambda_1}{\lambda_4}$, Chen and Zheng have also proved that

$$(1.4) \quad \frac{\lambda_2 + \lambda_3}{\lambda_1} \leq 5 + \frac{\lambda_1}{\lambda_4}.$$

For a general dimension $n \geq 2$, Ashbaugh and Benguria [4] (cf. Thompson [12]) proved

$$(1.5) \quad \frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \leq n + 4.$$

Furthermore, Ashbaugh and Benguria [4] (cf. Hile and Protter [9]) improved the result (1.5) to

$$(1.6) \quad \frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \leq n + 3 + \frac{\lambda_1}{\lambda_2}.$$

In this paper, by making use of the fact that eigenfunctions form an orthonormal basis of $L^2(\Omega)$ in place of the Rayleigh-Ritz formula, we obtain a general inequality for eigenvalues of the Laplacian. As an application, we study lower order eigenvalues of the Laplacian and obtain the following:

Theorem 1.1. *Let Ω be a bounded domain with a piecewise smooth boundary $\partial\Omega$ in an n -dimensional Euclidean space \mathbf{R}^n . Then, for $1 \leq i \leq n+2$, eigenvalues of the eigenvalue problem (1.1) satisfy at least one of the following:*

$$(1) \quad \frac{\lambda_2}{\lambda_1} < 2 - \frac{\lambda_1}{\lambda_i},$$

$$(2) \quad \frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \leq n + 3 + \frac{\lambda_1}{\lambda_i}.$$

Remark 1.1. Taking $i = 1$ in the theorem 1.1, we derive the result (1.5) of Ashbaugh and Benguria. Taking $i = 2$ in the theorem 1.1, we get the result (1.6) of Ashbaugh and Benguria. Taking $n = 2$, $i = 4$, we have the result (1.4) of Chen and Zheng [6].

2. MAIN RESULTS AND PROOFS

Let u_j be the orthonormal eigenfunction corresponding to the j -th eigenvalue λ_j , i.e. u_j satisfies

$$(2.1) \quad \begin{cases} \Delta u_j = -\lambda_j u_j & \text{in } \Omega, \\ u_j = 0 & \text{on } \partial\Omega, \\ \int_{\Omega} u_j u_k = \delta_{jk}. \end{cases}$$

In this section, first of all, by making use of the fact that $\{u_j\}_{j=1}^{\infty}$ forms an orthonormal basis of $L^2(\Omega)$ in place of the Rayleigh-Ritz formula, we obtain a general inequality for eigenvalues of the Laplacian.

Theorem 2.1. *Let Ω be a bounded domain with a piecewise smooth boundary $\partial\Omega$ in an n -dimensional Euclidean space \mathbf{R}^n . Then, there exists a Cartesian coordinate system (z_1, \cdots, z_n) of \mathbf{R}^n , such that, eigenvalues of the eigenvalue problem (1.1) satisfy*

$$(2.2) \quad \sum_{\alpha=1}^n \frac{\lambda_{k+1} - \lambda_1}{1 + \sum_{j=\alpha+1}^k (\lambda_{k+1} - \lambda_j) a_{\alpha j}^2} \leq 3\lambda_1 + \frac{\lambda_1^2}{\sigma_l},$$

where

$$a_{\alpha j} = \int_{\Omega} z_{\alpha} u_1 u_j, \quad \sigma_l = \lambda_1 + \frac{\lambda_{l+1} - \lambda_1}{1 + \sum_{j=2}^l \frac{\lambda_{l+1} - \lambda_j}{\lambda_j - \lambda_1} \left[1 - (\lambda_j - \lambda_1) \sum_{\alpha=1}^{j-1} a_{\alpha j}^2 \right]}.$$

Proof of Theorem 2.1. Let x_1, \dots, x_n be standard coordinate functions in \mathbf{R}^n . We consider the $n \times n$ -matrix $A = (A_{\alpha\beta})$ defined by

$$A_{\alpha\beta} = \int_{\Omega} x_{\alpha} u_1 u_{\beta+1}.$$

From the orthogonalization of Gram and Schmidt, there exist an upper triangle matrix $R = (R_{\alpha\beta})$ and an orthogonal matrix $Q = (q_{\alpha\beta})$ such that $R = QA$. Thus,

$$R_{\alpha\beta} = \sum_{\gamma=1}^n q_{\alpha\gamma} A_{\gamma\beta} = \int_{\Omega} \sum_{\gamma=1}^n q_{\alpha\gamma} x_{\gamma} u_1 u_{\beta+1} = 0, \text{ for } 1 \leq \beta < \alpha \leq n.$$

Defining $y_{\alpha} = \sum_{\gamma=1}^n q_{\alpha\gamma} x_{\gamma}$, we have

$$\int_{\Omega} y_{\alpha} u_1 u_{\beta+1} = \int_{\Omega} \sum_{\gamma=1}^n q_{\alpha\gamma} x_{\gamma} u_1 u_{\beta+1} = 0, \text{ for } 1 \leq \beta < \alpha \leq n.$$

Putting

$$z_{\alpha} = y_{\alpha} - y_{\alpha}^{(0)}, \quad y_{\alpha}^{(0)} = \int_{\Omega} y_{\alpha} u_1^2, \quad \text{for } 1 \leq \alpha \leq n$$

and

$$a_{\alpha j} = \int_{\Omega} z_{\alpha} u_1 u_j,$$

we have

$$(2.3) \quad a_{\alpha j} = 0, \quad \text{for } 1 \leq j \leq \alpha \leq n.$$

Defining

$$b_{\alpha j} = \int_{\Omega} u_j \nabla z_{\alpha} \cdot \nabla u_1,$$

from integration by parts, we obtain

$$\begin{aligned} -\lambda_j a_{\alpha j} &= \int_{\Omega} z_{\alpha} u_1 \Delta u_j = \int_{\Omega} \Delta(z_{\alpha} u_1) u_j \\ &= \int_{\Omega} \left(2 \nabla z_{\alpha} \cdot \nabla u_1 - \lambda_1 z_{\alpha} u_1 \right) u_j = 2b_{\alpha j} - \lambda_1 a_{\alpha j}, \end{aligned}$$

namely,

$$(2.4) \quad 2b_{\alpha j} = (\lambda_1 - \lambda_j) a_{\alpha j}.$$

Since $\{u_j\}_{j=1}^{\infty}$ is an orthonormal basis in $L^2(\Omega)$, we have

$$(2.5) \quad z_{\alpha} u_1 = \sum_{j=\alpha+1}^{\infty} a_{\alpha j} u_j \quad \text{and} \quad \|z_{\alpha} u_1\|^2 = \sum_{j=\alpha+1}^{\infty} a_{\alpha j}^2.$$

Furthermore,

$$(2.6) \quad 2 \int_{\Omega} z_{\alpha} u_1 \nabla z_{\alpha} \cdot \nabla u_1 = 2 \sum_{j=\alpha+1}^{\infty} a_{\alpha j} b_{\alpha j} = \sum_{j=\alpha+1}^{\infty} (\lambda_1 - \lambda_j) a_{\alpha j}^2.$$

On the other hand, from integration by parts, we get

$$-2 \int_{\Omega} z_{\alpha} u_1 \nabla z_{\alpha} \cdot \nabla u_1 = -\frac{1}{2} \int_{\Omega} \nabla z_{\alpha}^2 \cdot \nabla u_1^2 = \frac{1}{2} \int_{\Omega} u_1^2 \Delta z_{\alpha}^2 = 1.$$

Hence we have

$$(2.7) \quad \sum_{j=\alpha+1}^{\infty} (\lambda_j - \lambda_1) a_{\alpha j}^2 = 1.$$

For any positive integer k , we obtain

$$\begin{aligned} \sum_{j=\alpha+1}^{\infty} (\lambda_j - \lambda_1) a_{\alpha j}^2 &= \sum_{j=\alpha+1}^k (\lambda_j - \lambda_1) a_{\alpha j}^2 + \sum_{j=k+1}^{\infty} (\lambda_j - \lambda_1) a_{\alpha j}^2 \\ &\geq \sum_{j=\alpha+1}^k (\lambda_j - \lambda_1) a_{\alpha j}^2 + (\lambda_{k+1} - \lambda_1) \sum_{j=k+1}^{\infty} a_{\alpha j}^2 \\ &= \sum_{j=\alpha+1}^k (\lambda_j - \lambda_1) a_{\alpha j}^2 + (\lambda_{k+1} - \lambda_1) \sum_{j=\alpha+1}^{\infty} a_{\alpha j}^2 - (\lambda_{k+1} - \lambda_1) \sum_{j=\alpha+1}^k a_{\alpha j}^2 \\ &= \sum_{j=\alpha+1}^k (\lambda_j - \lambda_{k+1}) a_{\alpha j}^2 + (\lambda_{k+1} - \lambda_1) \sum_{j=\alpha+1}^{\infty} a_{\alpha j}^2. \end{aligned}$$

Thus, we infer

$$(2.8) \quad (\lambda_{k+1} - \lambda_1) \|z_{\alpha} u_1\|^2 \leq 1 + \sum_{j=\alpha+1}^k (\lambda_{k+1} - \lambda_j) a_{\alpha j}^2.$$

For some real constant $t > \frac{1}{2}$, from integration by parts, we get

$$\begin{aligned} \int_{\Omega} |\nabla u_1^{t-1}|^2 u_1^2 &= (t-1)^2 \int_{\Omega} u_1^{2t-2} |\nabla u_1|^2 \\ &= \frac{(t-1)^2}{2t-1} \int_{\Omega} \nabla u_1 \cdot \nabla u_1^{2t-1} \\ (2.9) \quad &= -\frac{(t-1)^2}{2t-1} \int_{\Omega} u_1^{2t-1} \Delta u_1 \\ &= \frac{(t-1)^2}{2t-1} \lambda_1 \int_{\Omega} u_1^{2t}. \end{aligned}$$

Letting

$$d_j = \int_{\Omega} u_1^t u_j,$$

we know

$$(2.10) \quad u_1^t = \sum_{j=1}^{\infty} d_j u_j, \quad \|u_1^t\|^2 = \int_{\Omega} u_1^{2t} = \sum_{j=1}^{\infty} d_j^2.$$

Since for any function $f \in C^2(\Omega) \cap C(\bar{\Omega})$, from integration by parts, we have

$$(2.11) \quad -2 \int_{\Omega} f u_1 \nabla f \cdot \nabla u_1 = \int_{\Omega} u_1^2 f \Delta f + \int_{\Omega} |\nabla f|^2 u_1^2.$$

Taking $f = u_1^{t-1}$ in (2.11), we get

$$\begin{aligned} \int_{\Omega} |\nabla u_1^{t-1}|^2 u_1^2 &= -2 \int_{\Omega} u_1^t \nabla u_1^{t-1} \cdot \nabla u_1 - \int_{\Omega} u_1^{t+1} \Delta u_1^{t-1} \\ &= -\sum_{j=1}^{\infty} d_j \left(2 \int_{\Omega} u_j \nabla u_1^{t-1} \cdot \nabla u_1 + \int_{\Omega} u_j u_1 \Delta u_1^{t-1} \right) \\ &= -\sum_{j=1}^{\infty} d_j \left(\int_{\Omega} u_j \Delta u_1^t - \int_{\Omega} u_j u_1^{t-1} \Delta u_1 \right) \\ &= -\sum_{j=1}^{\infty} d_j \left(\int_{\Omega} u_1^t \Delta u_j - \int_{\Omega} u_j u_1^{t-1} \Delta u_1 \right) \\ &= \sum_{j=1}^{\infty} d_j \left(\lambda_j \int_{\Omega} u_1^t u_j - \lambda_1 \int_{\Omega} u_j u_1^t \right) \\ &= \sum_{j=2}^{\infty} (\lambda_j - \lambda_1) d_j^2. \end{aligned}$$

Thus, by (2.9), we get

$$(2.12) \quad \int_{\Omega} |\nabla u_1^{t-1}|^2 u_1^2 = \frac{(t-1)^2}{2t-1} \lambda_1 \int_{\Omega} u_1^{2t} = \sum_{j=2}^{\infty} (\lambda_j - \lambda_1) d_j^2.$$

Taking

$$(2.13) \quad \beta_j = \frac{d_j}{\sqrt{\frac{(t-1)^2}{2t-1} \lambda_1 \int_{\Omega} u_1^{2t}}},$$

then, we have

$$(2.14) \quad \sum_{j=2}^{\infty} (\lambda_j - \lambda_1) \beta_j^2 = 1.$$

For any positive integer l , we obtain

$$\begin{aligned} \sum_{j=2}^{\infty} (\lambda_j - \lambda_1) \beta_j^2 &= \sum_{j=2}^l (\lambda_j - \lambda_1) \beta_j^2 + \sum_{j=l+1}^{\infty} (\lambda_j - \lambda_1) \beta_j^2 \\ &\geq \sum_{j=2}^l (\lambda_j - \lambda_1) \beta_j^2 + (\lambda_{l+1} - \lambda_1) \sum_{j=l+1}^{\infty} \beta_j^2 \\ &= \sum_{j=2}^l (\lambda_j - \lambda_1) \beta_j^2 + (\lambda_{l+1} - \lambda_1) \sum_{j=2}^{\infty} \beta_j^2 - (\lambda_{l+1} - \lambda_1) \sum_{j=2}^l \beta_j^2 \\ &= \sum_{j=2}^l (\lambda_j - \lambda_{l+1}) \beta_j^2 + (\lambda_{l+1} - \lambda_1) \sum_{j=2}^{\infty} \beta_j^2. \end{aligned}$$

Thus, we infer

$$(2.15) \quad (\lambda_{l+1} - \lambda_1) \sum_{j=2}^{\infty} \beta_j^2 \leq 1 + \sum_{j=2}^l (\lambda_{l+1} - \lambda_j) \beta_j^2.$$

Set

$$B(t) = \frac{\int_{\Omega} u_1^{2t}}{(\int_{\Omega} u_1^{t+1})^2}.$$

From (2.10), (2.12), (2.13) and (2.14), we infer

$$(2.16) \quad \frac{(t-1)^2}{2t-1} \lambda_1 \sum_{j=1}^{\infty} \beta_j^2 = 1.$$

Since

$$\left(\int_{\Omega} u_1^{t+1} \right)^2 = d_1^2 = \frac{(t-1)^2}{2t-1} \lambda_1 \beta_1^2 \int_{\Omega} u_1^{2t},$$

according to the definition of $B(t)$, we have

$$(2.17) \quad B(t) = \frac{\int_{\Omega} u_1^{2t}}{\left(\int_{\Omega} u_1^{t+1} \right)^2} = \frac{1}{\frac{(t-1)^2}{2t-1} \lambda_1 \beta_1^2} = \frac{1}{1 - \frac{(t-1)^2}{2t-1} \lambda_1 \sum_{j=2}^{\infty} \beta_j^2}.$$

By integration by parts, we have

$$d_1 = \int_{\Omega} u_1^{t+1} = \frac{1}{2} \int_{\Omega} u_1^{t+1} \Delta z_{\alpha}^2 = -(t+1) \int_{\Omega} z_{\alpha} u_1^t \nabla z_{\alpha} \cdot \nabla u_1.$$

From the Cauchy-Schwarz inequality, we get

$$(2.18) \quad d_1^2 \leq (t+1)^2 \int_{\Omega} (z_{\alpha} u_1)^2 \int_{\Omega} u_1^{2t-2} (\nabla z_{\alpha} \cdot \nabla u_1)^2.$$

Hence, we have

$$(2.19) \quad \sum_{\alpha=1}^n \frac{1}{\|z_{\alpha} u_1\|^2} \leq \frac{(t+1)^2 \int_{\Omega} u_1^{2t-2} |\nabla u_1|^2}{(\int_{\Omega} u_1^{t+1})^2} = \frac{(t+1)^2}{2t-1} \lambda_1 B(t).$$

For any fixed $j \geq 2$, we choose an orthonormal transformation $\tilde{z}_{\alpha} = \sum_{\beta=1}^n h_{\alpha\beta} z_{\beta}$ ($1 \leq \alpha \leq n$) with $\det(h_{\alpha\beta}) = 1$ such that

$$\tilde{a}_{\alpha j} = \int_{\Omega} \tilde{z}_{\alpha} u_1 u_j = \begin{cases} \sqrt{\sum_{\beta=1}^{j-1} a_{\beta j}^2}, & \alpha = 1, \\ 0, & \alpha \geq 1. \end{cases}$$

From the definition of \tilde{z}_α and (2.4), we get

$$\begin{aligned}
0 &= -\frac{2}{t+1} \int_{\Omega} u_1^{t+1} \Delta \tilde{z}_\alpha = 2 \int_{\Omega} u_1^t \nabla \tilde{z}_\alpha \cdot \nabla u_1 \\
&= 2 \sum_{p=1}^{\infty} d_p \int_{\Omega} u_p \nabla \tilde{z}_\alpha \cdot \nabla u_1 \\
&= 2 \sum_{p=1}^{\infty} \sum_{\beta=1}^n d_p h_{\alpha\beta} b_{\beta p} \\
&= \sum_{p=1}^{\infty} \sum_{\beta=1}^n d_p h_{\alpha\beta} (\lambda_1 - \lambda_p) a_{\beta p} \\
&= \sum_{p=1}^{\infty} (\lambda_1 - \lambda_p) d_p \tilde{a}_{\alpha p}.
\end{aligned}$$

Hence, from (2.13), we obtain

$$(2.20) \quad \sum_{p=2}^{\infty} (\lambda_p - \lambda_1) \beta_p \tilde{a}_{\alpha p} = 0.$$

Notice that

$$\begin{aligned}
1 &= -2 \int_{\Omega} \tilde{z}_\alpha u_1 \nabla \tilde{z}_\alpha \cdot \nabla u_1 \\
&= -2 \sum_{p=1}^{\infty} \tilde{a}_{\alpha p} h_{\alpha\beta} b_{\beta p} \\
(2.21) \quad &= - \sum_{p=1}^{\infty} \tilde{a}_{\alpha p} h_{\alpha\beta} (\lambda_1 - \lambda_p) a_{\beta p} \\
&= \sum_{p=2}^{\infty} (\lambda_p - \lambda_1) \tilde{a}_{\alpha p}^2.
\end{aligned}$$

Thus, from (2.20) and the Cauchy-Schwarz inequality, we infer

$$\left((\lambda_j - \lambda_1) \beta_j \tilde{a}_{\alpha j} \right)^2 \leq \left(\sum_{p=2, p \neq j}^{\infty} (\lambda_p - \lambda_1) \beta_p^2 \right) \left(\sum_{p=2, p \neq j}^{\infty} (\lambda_p - \lambda_1) \tilde{a}_{\alpha p}^2 \right).$$

Then, according to (2.21) and (2.14), we derive

$$(\lambda_j - \lambda_1)^2 \beta_j^2 \tilde{a}_{\alpha j}^2 \leq \left(1 - (\lambda_j - \lambda_1) \beta_j^2 \right) \left(1 - (\lambda_j - \lambda_1) \tilde{a}_{\alpha j}^2 \right).$$

Hence, we have

$$(\lambda_j - \lambda_1) \beta_j^2 + (\lambda_j - \lambda_1) \tilde{a}_{\alpha j}^2 \leq 1,$$

namely,

$$(2.22) \quad (\lambda_j - \lambda_1) \beta_j^2 + (\lambda_j - \lambda_1) \sum_{\beta=1}^{j-1} a_{\beta j}^2 \leq 1.$$

From (2.8), (2.15), (2.17), (2.19) and (2.22), we have

$$\begin{aligned}
& \sum_{\alpha=1}^n \frac{\lambda_{k+1} - \lambda_1}{1 + \sum_{j=\alpha+1}^k (\lambda_{k+1} - \lambda_j) a_{\alpha j}^2} \\
& \leq \sum_{\alpha=1}^n \frac{1}{\|z_{\alpha} u_1\|^2} \\
& \leq \frac{(t+1)^2}{2t-1} \lambda_1 B(t) \\
& = \frac{(t+1)^2}{2t-1} \frac{\lambda_1}{1 - \frac{(t-1)^2}{2t-1} \lambda_1 \sum_{j=2}^{\infty} \beta_j^2} \\
& \leq \frac{(t+1)^2}{2t-1} \frac{\lambda_1}{1 - \frac{(t-1)^2}{2t-1} \frac{\lambda_1}{\lambda_{l+1} - \lambda_1} \left(1 + \sum_{j=2}^l (\lambda_{l+1} - \lambda_j) \beta_j^2\right)} \\
& \leq \frac{(t+1)^2}{2t-1} \frac{\lambda_1}{1 - \frac{(t-1)^2}{2t-1} \frac{\lambda_1}{\lambda_{l+1} - \lambda_1} \left(1 + \sum_{j=2}^l \frac{\lambda_{l+1} - \lambda_j}{\lambda_j - \lambda_1} \left[1 - (\lambda_j - \lambda_1) \sum_{\alpha=1}^{j-1} a_{\alpha j}^2\right]\right)} \\
& = \frac{(t+1)^2}{2t-1} \frac{\lambda_1}{1 - \frac{(t-1)^2}{2t-1} \frac{\lambda_1}{\sigma_l - \lambda_1}}.
\end{aligned}$$

Taking $t = \frac{2\sigma_l}{\sigma_l + \lambda_1}$, we obtain (2.2). \square

In order to prove the theorem 1.1, we prepare the following lemmas.

Lemma 2.1. *Let $\{\theta_i\}_{i=1}^{m+2}$ be an increasing real sequence and let $\omega = (\omega_{jk})$ be a real $(m+1) \times (m+1)$ -matrix. Then the following equality holds:*

$$\begin{aligned}
& \sum_{i=1}^m \frac{\theta_{m+2} - \theta_1}{1 + \sum_{p=i+1}^{m+1} (\theta_{m+2} - \theta_p) \omega_{ip}^2} - \sum_{i=1}^m (\theta_{i+1} - \theta_1) \\
& = \sum_{j=1}^m \sum_{i=1}^{m+1-j} \frac{(\theta_{i+j+1} - \theta_{i+j}) \left[1 - \sum_{p=i+1}^{i+j} (\theta_p - \theta_1) \omega_{ip}^2\right]}{\left[1 + \sum_{p=i+1}^{i+j-1} (\theta_{i+j} - \theta_p) \omega_{ip}^2\right] \left[1 + \sum_{p=i+1}^{i+j} (\theta_{i+j+1} - \theta_p) \omega_{ip}^2\right]}.
\end{aligned}$$

Proof. For $1 \leq j \leq m+1$, we define

$$F_j = \sum_{i=1}^{m+1-j} \frac{(\theta_{m+2} - \theta_{i+j}) \left[1 - \sum_{p=i+1}^{i+j} (\theta_p - \theta_1) \omega_{ip}^2\right] - (\theta_{i+j} - \theta_1) \sum_{p=i+j+1}^{m+1} (\theta_{m+2} - \theta_p) \omega_{ip}^2}{\left[1 + \sum_{p=i+1}^{i+j-1} (\theta_{i+j} - \theta_p) \omega_{ip}^2\right] \left[1 + \sum_{p=i+1}^{m+1} (\theta_{m+2} - \theta_p) \omega_{ip}^2\right]},$$

$$G_j = \sum_{i=1}^{m+1-j} \frac{(\theta_{i+j+1} - \theta_{i+j}) \left[1 - \sum_{p=i+1}^{i+j} (\theta_p - \theta_1) \omega_{ip}^2 \right]}{\left[1 + \sum_{p=i+1}^{i+j-1} (\theta_{i+j} - \theta_p) \omega_{ip}^2 \right] \left[1 + \sum_{p=i+1}^{i+j} (\theta_{i+j+1} - \theta_p) \omega_{ip}^2 \right]},$$

$$D_{ij} = \left[1 + \sum_{p=i+1}^{i+j-1} (\theta_{i+j} - \theta_p) \omega_{ip}^2 \right] \left[1 + \sum_{p=i+1}^{i+j} (\theta_{i+j+1} - \theta_p) \omega_{ip}^2 \right] \left[1 + \sum_{p=i+1}^{m+1} (\theta_{m+2} - \theta_p) \omega_{ip}^2 \right].$$

Then, we have the following recursion formula:

$$(2.23) \quad F_j - G_j = F_{j+1}.$$

In fact,

$$\begin{aligned} & F_j - G_j \\ &= \sum_{i=1}^{m+1-j} \frac{1}{D_{ij}} \left\{ (\theta_{m+2} - \theta_{i+j}) \left[1 - \sum_{p=i+1}^{i+j} (\theta_p - \theta_1) \omega_{ip}^2 \right] \left[1 + \sum_{p=i+1}^{i+j} (\theta_{i+j+1} - \theta_p) \omega_{ip}^2 \right] \right. \\ & \quad - (\theta_{i+j} - \theta_1) \sum_{p=i+j+1}^{m+1} (\theta_{m+2} - \theta_p) \omega_{ip}^2 \left[1 + \sum_{p=i+1}^{i+j} (\theta_{i+j+1} - \theta_p) \omega_{ip}^2 \right] \\ & \quad \left. - (\theta_{i+j+1} - \theta_{i+j}) \left[1 - \sum_{p=i+1}^{i+j} (\theta_p - \theta_1) \omega_{ip}^2 \right] \left[1 + \sum_{p=i+1}^{m+1} (\theta_{m+2} - \theta_p) \omega_{ip}^2 \right] \right\} \\ &= \sum_{i=1}^{m+1-j} \frac{1}{D_{ij}} \left\{ \left[1 - \sum_{p=i+1}^{i+j} (\theta_p - \theta_1) \omega_{ip}^2 \right] \left[(\theta_{m+2} - \theta_{i+j}) + (\theta_{m+2} - \theta_{i+j}) \sum_{p=i+1}^{i+j} (\theta_{i+j+1} - \theta_p) \omega_{ip}^2 \right. \right. \\ & \quad \left. - (\theta_{i+j+1} - \theta_{i+j}) - (\theta_{i+j+1} - \theta_{i+j}) \sum_{p=i+1}^{m+1} (\theta_{m+2} - \theta_p) \omega_{ip}^2 \right] \\ & \quad \left. - (\theta_{i+j} - \theta_1) \sum_{p=i+j+1}^{m+1} (\theta_{m+2} - \theta_p) \omega_{ip}^2 \left[1 + \sum_{p=i+1}^{i+j} (\theta_{i+j+1} - \theta_p) \omega_{ip}^2 \right] \right\} \\ &= \sum_{i=1}^{m+1-j} \frac{1}{D_{ij}} \left\{ \left[1 - \sum_{p=i+1}^{i+j} (\theta_p - \theta_1) \omega_{ip}^2 \right] \left[(\theta_{m+2} - \theta_{i+j+1}) + \sum_{p=i+1}^{i+j} (\theta_{m+2} - \theta_{i+j}) (\theta_{i+j+1} - \theta_p) \omega_{ip}^2 \right. \right. \\ & \quad \left. - \sum_{p=i+1}^{i+j} (\theta_{i+j+1} - \theta_{i+j}) (\theta_{m+2} - \theta_p) \omega_{ip}^2 - (\theta_{i+j+1} - \theta_{i+j}) \sum_{p=i+j+1}^{m+1} (\theta_{m+2} - \theta_p) \omega_{ip}^2 \right] \\ & \quad \left. - (\theta_{i+j} - \theta_1) \sum_{p=i+j+1}^{m+1} (\theta_{m+2} - \theta_p) \omega_{ip}^2 \left[1 + \sum_{p=i+1}^{i+j} (\theta_{i+j+1} - \theta_p) \omega_{ip}^2 \right] \right\} \\ &= \sum_{i=1}^{m+1-j} \frac{1}{D_{ij}} \left\{ \left[1 - \sum_{p=i+1}^{i+j} (\theta_p - \theta_1) \omega_{ip}^2 \right] \left[(\theta_{m+2} - \theta_{i+j+1}) + (\theta_{m+2} - \theta_{i+j+1}) \sum_{p=i+1}^{i+j} (\theta_{i+j} - \theta_p) \omega_{ip}^2 \right. \right. \\ & \quad \left. - (\theta_{i+j+1} - \theta_{i+j}) \sum_{p=i+j+1}^{m+1} (\theta_{m+2} - \theta_p) \omega_{ip}^2 \right] - (\theta_{i+j} - \theta_1) \sum_{p=i+j+1}^{m+1} (\theta_{m+2} - \theta_p) \omega_{ip}^2 \end{aligned}$$

$$\begin{aligned}
& -(\theta_{i+j} - \theta_1) \sum_{p=i+j+1}^{m+1} (\theta_{m+2} - \theta_p) \omega_{ip}^2 \sum_{p=i+1}^{i+j} (\theta_{i+j+1} - \theta_p) \omega_{ip}^2 \Big\} \\
&= \sum_{i=1}^{m+1-j} \frac{1}{D_{ij}} \left\{ (\theta_{m+2} - \theta_{i+j+1}) \left[1 - \sum_{p=i+1}^{i+j} (\theta_p - \theta_1) \omega_{ip}^2 \right] \left[1 + \sum_{p=i+1}^{i+j-1} (\theta_{i+j} - \theta_p) \omega_{ip}^2 \right] \right. \\
&\quad - (\theta_{i+j+1} - \theta_{i+j}) \sum_{p=i+j+1}^{m+1} (\theta_{m+2} - \theta_p) \omega_{ip}^2 \left[1 - \sum_{p=i+1}^{i+j} (\theta_p - \theta_1) \omega_{ip}^2 \right] \\
&\quad - (\theta_{i+j} - \theta_1) \sum_{p=i+j+1}^{m+1} (\theta_{m+2} - \theta_p) \omega_{ip}^2 \\
&\quad \left. - (\theta_{i+j} - \theta_1) \sum_{p=i+j+1}^{m+1} (\theta_{m+2} - \theta_p) \omega_{ip}^2 \sum_{p=i+1}^{i+j} (\theta_{i+j+1} - \theta_p) \omega_{ip}^2 \right\} \\
&= \sum_{i=1}^{m+1-j} \frac{1}{D_{ij}} \left\{ (\theta_{m+2} - \theta_{i+j+1}) \left[1 - \sum_{p=i+1}^{i+j} (\theta_p - \theta_1) \omega_{ip}^2 \right] \left[1 + \sum_{p=i+1}^{i+j-1} (\theta_{i+j} - \theta_p) \omega_{ip}^2 \right] \right. \\
&\quad \left. - (\theta_{i+j+1} - \theta_1) \sum_{p=i+j+1}^{m+1} (\theta_{m+2} - \theta_p) \omega_{ip}^2 \left[1 + \sum_{p=i+1}^{i+j} (\theta_{i+j} - \theta_p) \omega_{ip}^2 \right] \right\} \\
&= \sum_{i=1}^{m+1-j} \frac{\left[1 + \sum_{p=i+1}^{i+j-1} (\theta_{i+j} - \theta_p) \omega_{ip}^2 \right]}{D_{ij}} \left\{ (\theta_{m+2} - \theta_{i+j+1}) \left[1 - \sum_{p=i+1}^{i+j} (\theta_p - \theta_1) \omega_{ip}^2 \right] \right. \\
&\quad \left. - (\theta_{i+j+1} - \theta_1) \sum_{p=i+j+1}^{m+1} (\theta_{m+2} - \theta_p) \omega_{ip}^2 \right\} \\
&= \sum_{i=1}^{m-j} \frac{(\theta_{m+2} - \theta_{i+j+1}) \left[1 - \sum_{p=i+1}^{i+j+1} (\theta_p - \theta_1) \omega_{ip}^2 \right] - (\theta_{i+j+1} - \theta_1) \sum_{p=i+j+2}^{m+1} (\theta_{m+2} - \theta_p) \omega_{ip}^2}{\left[1 + \sum_{p=i+1}^{i+j} (\theta_{i+j+1} - \theta_p) \omega_{ip}^2 \right] \left[1 + \sum_{p=i+1}^{m+1} (\theta_{m+2} - \theta_p) \omega_{ip}^2 \right]} \\
&= F_{j+1}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& \sum_{i=1}^m \frac{\theta_{m+2} - \theta_1}{1 + \sum_{p=i+1}^{m+1} (\theta_{m+2} - \theta_p) \omega_{ip}^2} - \sum_{i=1}^m (\theta_{i+1} - \theta_1) \\
&= \sum_{i=1}^m \frac{(\theta_{m+2} - \theta_{i+1}) \left[1 - (\theta_{i+1} - \theta_1) \omega_{i+1}^2 \right] - (\theta_{i+1} - \theta_1) \sum_{p=i+2}^{m+1} (\theta_{m+2} - \theta_p) \omega_{ip}^2}{1 + \sum_{p=i+1}^{m+1} (\theta_{m+2} - \theta_p) \omega_{ip}^2} \\
&= F_1 = G_1 + F_2 = G_1 + G_2 + F_3 = \cdots = \sum_{j=1}^m G_j + F_{m+1}.
\end{aligned}$$

Since $F_{m+1} = 0$, we complete the proof of the lemma 2.1. \square

Lemma 2.2. *Let $\{\theta_i\}_{i=1}^{m+2}$ be an increasing real sequence and let $\omega = (\omega_{jk})$ be a real $(m+1) \times (m+1)$ -matrix. Then the following equality holds:*

$$\begin{aligned} & \sum_{j=1}^m \sum_{i=1}^{m+1-j} \frac{(\theta_{i+j+1} - \theta_{i+j})(\theta_2 - \theta_1)^2}{(\theta_{i+j+1} - \theta_1)(\theta_{i+j} - \lambda_1)} \left[1 - \sum_{p=i+1}^{i+j} (\theta_p - \theta_1) \omega_{ip}^2 \right] \\ &= \sum_{j=2}^{m+1} \frac{(\theta_{j+1} - \theta_j)(\theta_2 - \theta_1)^2}{(\theta_{j+1} - \theta_1)(\theta_j - \lambda_1)} \sum_{i=1}^{j-1} \left[1 - \sum_{p=i+1}^j (\theta_p - \theta_1) \omega_{ip}^2 \right]. \end{aligned}$$

Proof. In fact, we have

$$\begin{aligned} & \sum_{j=1}^m \sum_{i=1}^{m+1-j} \frac{(\theta_{i+j+1} - \theta_{i+j})(\theta_2 - \theta_1)^2}{(\theta_{i+j+1} - \theta_1)(\theta_{i+j} - \lambda_1)} \left[1 - \sum_{p=i+1}^{i+j} (\theta_p - \theta_1) \omega_{ip}^2 \right] \\ &= \frac{(\theta_{m+2} - \theta_{m+1})(\theta_2 - \theta_1)^2}{(\theta_{m+2} - \theta_1)(\theta_{m+1} - \lambda_1)} \left[1 - \sum_{p=2}^{m+1} (\theta_p - \theta_1) \omega_{1p}^2 \right] \\ &+ \sum_{i=1}^2 \frac{(\theta_{i+m} - \theta_{i+m-1})(\theta_2 - \theta_1)^2}{(\theta_{i+m} - \theta_1)(\theta_{i+m-1} - \lambda_1)} \left[1 - \sum_{p=i+1}^{i+m-1} (\theta_p - \theta_1) \omega_{ip}^2 \right] \\ &+ \cdots + \sum_{i=1}^{m-1} \frac{(\theta_{i+3} - \theta_{i+2})(\theta_2 - \theta_1)^2}{(\theta_{i+3} - \theta_1)(\theta_{i+2} - \lambda_1)} \left[1 - \sum_{p=i+1}^{i+2} (\theta_p - \theta_1) \omega_{ip}^2 \right] \\ &+ \sum_{i=1}^m \frac{(\theta_{i+2} - \theta_{i+1})(\theta_2 - \theta_1)^2}{(\theta_{i+2} - \theta_1)(\theta_{i+1} - \lambda_1)} \left[1 - (\theta_{i+1} - \theta_1) \omega_{ii+1}^2 \right] \\ &= \frac{(\theta_{m+2} - \theta_{m+1})(\theta_2 - \theta_1)^2}{(\theta_{m+2} - \theta_1)(\theta_{m+1} - \lambda_1)} \sum_{i=1}^m \left[1 - \sum_{p=i+1}^{m+1} (\theta_p - \theta_1) \omega_{ip}^2 \right] \\ &+ \frac{(\theta_{m+1} - \theta_m)(\theta_2 - \theta_1)^2}{(\theta_{m+1} - \theta_1)(\theta_m - \lambda_1)} \sum_{i=1}^{m-1} \left[1 - \sum_{p=i+1}^m (\theta_p - \theta_1) \omega_{ip}^2 \right] \\ &+ \cdots + \frac{(\theta_3 - \theta_2)(\theta_2 - \theta_1)^2}{(\theta_3 - \theta_1)(\theta_2 - \lambda_1)} \left[1 - (\theta_2 - \theta_1) \omega_{12}^2 \right] \\ &= \sum_{j=2}^{m+1} \frac{(\theta_{j+1} - \theta_j)(\theta_2 - \theta_1)^2}{(\theta_{j+1} - \theta_1)(\theta_j - \lambda_1)} \sum_{i=1}^{j-1} \left[1 - \sum_{p=i+1}^j (\theta_p - \theta_1) \omega_{ip}^2 \right]. \end{aligned}$$

\square

Lemma 2.3. *Let $\{\theta_i\}_{i=1}^{m+2}$ be an increasing real sequence and let $\omega = (\omega_{jk})$ be a real $(m+1) \times (m+1)$ -matrix. Then, for $1 \leq i \leq m+2$, the following equality holds:*

$$\begin{aligned} & \sum_{j=2}^{i-1} \frac{\theta_i - \theta_j}{\theta_j - \theta_1} \left[1 - (\theta_j - \theta_1) \sum_{k=1}^{j-1} \omega_{kj}^2 \right] \\ &= \sum_{j=2}^{i-1} \frac{(\theta_i - \theta_1)(\theta_{j+1} - \theta_j)}{(\theta_{j+1} - \theta_1)(\theta_j - \theta_1)} \sum_{k=1}^{j-1} \left[1 - \sum_{p=k+1}^j (\theta_p - \theta_1) \omega_{kp}^2 \right]. \end{aligned}$$

Proof. For $1 \leq i \leq m+2$, we have

$$\begin{aligned}
& \sum_{j=2}^{i-1} \frac{\theta_i - \theta_j}{\theta_j - \theta_1} \left[1 - (\theta_j - \theta_1) \sum_{k=1}^{j-1} \omega_{kj}^2 \right] \\
&= \sum_{j=2}^{i-1} \frac{\theta_i - \theta_j}{\theta_j - \theta_1} \sum_{k=1}^{j-1} \left[1 - \sum_{p=k+1}^j (\theta_p - \theta_1) \omega_{kp}^2 \right] - \sum_{j=3}^{i-1} \frac{\theta_i - \theta_j}{\theta_j - \theta_1} \sum_{k=1}^{j-2} \left[1 - \sum_{p=k+1}^{j-1} (\theta_p - \theta_1) \omega_{kp}^2 \right] \\
&= \sum_{j=2}^{i-1} \frac{\theta_i - \theta_j}{\theta_j - \theta_1} \sum_{k=1}^{j-1} \left[1 - \sum_{p=k+1}^j (\theta_p - \theta_1) \omega_{kp}^2 \right] - \sum_{j=2}^{i-2} \frac{\theta_i - \theta_{j+1}}{\theta_{j+1} - \theta_1} \sum_{k=1}^{j-1} \left[1 - \sum_{p=k+1}^j (\theta_p - \theta_1) \omega_{kp}^2 \right] \\
&= \frac{\theta_i - \theta_{i-1}}{\theta_{i-1} - \theta_1} \sum_{k=1}^{i-2} \left[1 - \sum_{p=k+1}^{i-1} (\theta_p - \theta_1) \omega_{kp}^2 \right] + \sum_{j=2}^{i-2} \frac{(\theta_i - \theta_1)(\theta_{j+1} - \theta_j)}{(\theta_{j+1} - \theta_1)(\theta_j - \theta_1)} \sum_{k=1}^{j-1} \left[1 - \sum_{p=k+1}^j (\theta_p - \theta_1) \omega_{kp}^2 \right] \\
&= \sum_{j=2}^{i-1} \frac{(\theta_i - \theta_1)(\theta_{j+1} - \theta_j)}{(\theta_{j+1} - \theta_1)(\theta_j - \theta_1)} \sum_{k=1}^{j-1} \left[1 - \sum_{p=k+1}^j (\theta_p - \theta_1) \omega_{kp}^2 \right].
\end{aligned}$$

□

Proof of Theorem 1.1. If $\frac{\lambda_2}{\lambda_1} < 2 - \frac{\lambda_1}{\lambda_i}$, we know that the theorem 1.1 is proved. Hence, we assume $\frac{\lambda_2}{\lambda_1} \geq 2 - \frac{\lambda_1}{\lambda_i}$. Taking $k = n+1$, $l = i-1$ in (2.2), we have

$$(2.24) \quad \sum_{\alpha=1}^n (\lambda_{\alpha+1} - \lambda_1) + B \leq 3\lambda_1 + \frac{\lambda_1^2}{\lambda_i} + C,$$

where

$$\begin{aligned}
B &= \sum_{\alpha=1}^n \frac{\lambda_{n+2} - \lambda_1}{1 + \sum_{j=\alpha+1}^{n+1} (\lambda_{n+2} - \lambda_j) a_{\alpha j}^2} - \sum_{\alpha=1}^n (\lambda_{\alpha+1} - \lambda_1), \\
C &= \frac{(\lambda_i - \lambda_1) \sum_{j=2}^{i-1} \frac{\lambda_i - \lambda_j}{\lambda_j - \lambda_1} \left[1 - (\lambda_j - \lambda_1) \sum_{\alpha=1}^{j-1} a_{\alpha j}^2 \right]}{\frac{\lambda_i}{\lambda_1} \left\{ \frac{\lambda_i}{\lambda_1} + \sum_{j=2}^{i-1} \frac{\lambda_i - \lambda_j}{\lambda_j - \lambda_1} \left[1 - (\lambda_j - \lambda_1) \sum_{\alpha=1}^{j-1} a_{\alpha j}^2 \right] \right\}}.
\end{aligned}$$

It follows from the lemma 2.1 that

$$(2.25) \quad B = \sum_{\beta=1}^n \sum_{\alpha=1}^{n+1-\beta} \frac{(\lambda_{\alpha+\beta+1} - \lambda_{\alpha+\beta}) \left[1 - \sum_{p=\alpha+1}^{\alpha+\beta} (\lambda_p - \lambda_1) a_{\alpha p}^2 \right]}{\left[1 + \sum_{p=\alpha+1}^{\alpha+\beta-1} (\lambda_{\alpha+\beta} - \lambda_p) a_{\alpha p}^2 \right] \cdot \left[1 + \sum_{p=\alpha+1}^{\alpha+\beta} (\lambda_{\alpha+\beta+1} - \lambda_p) a_{\alpha p}^2 \right]}.$$

For any positive integer γ , from (2.7), we can get

$$\sum_{p=\alpha+1}^{\gamma} (\lambda_p - \lambda_1) a_{\alpha p}^2 \leq 1.$$

Then, we have the following recursion formula:

$$\begin{aligned}
& 1 + \sum_{p=\alpha+1}^{\gamma} (\lambda_{\gamma+1} - \lambda_p) a_{\alpha p}^2 \\
&= 1 + \frac{\lambda_{\gamma+1} - \lambda_{\gamma}}{\lambda_{\gamma} - \lambda_1} \sum_{p=\alpha+1}^{\gamma} (\lambda_p - \lambda_1) a_{\alpha p}^2 - \frac{\lambda_{\gamma+1} - \lambda_{\gamma}}{\lambda_{\gamma} - \lambda_1} \sum_{p=\alpha+1}^{\gamma-1} (\lambda_p - \lambda_1) a_{\alpha p}^2 + \sum_{p=\alpha+1}^{\gamma-1} (\lambda_{\gamma+1} - \lambda_p) a_{\alpha p}^2 \\
&\leq 1 + \frac{\lambda_{\gamma+1} - \lambda_{\gamma}}{\lambda_{\gamma} - \lambda_1} - \frac{\lambda_{\gamma+1} - \lambda_{\gamma}}{\lambda_{\gamma} - \lambda_1} \sum_{p=\alpha+1}^{\gamma-1} (\lambda_p - \lambda_1) a_{\alpha p}^2 + \sum_{p=\alpha+1}^{\gamma-1} (\lambda_{\gamma+1} - \lambda_p) a_{\alpha p}^2 \\
&= \frac{\lambda_{\gamma+1} - \lambda_1}{\lambda_{\gamma} - \lambda_1} \left[1 + \sum_{p=\alpha+1}^{\gamma-1} (\lambda_{\gamma} - \lambda_p) a_{\alpha p}^2 \right].
\end{aligned}$$

Therefore, we can obtain

$$\begin{aligned}
& 1 + \sum_{p=\alpha+1}^{\alpha+\beta} (\lambda_{\alpha+\beta+1} - \lambda_p) a_{\alpha p}^2 \\
&\leq \frac{\lambda_{\alpha+\beta+1} - \lambda_1}{\lambda_{\alpha+\beta} - \lambda_1} \cdot \frac{\lambda_{\alpha+\beta} - \lambda_1}{\lambda_{\alpha+\beta-1} - \lambda_1} \cdots \frac{\lambda_{\alpha+2} - \lambda_1}{\lambda_{\alpha+1} - \lambda_1} = \frac{\lambda_{\alpha+\beta+1} - \lambda_1}{\lambda_{\alpha+1} - \lambda_1}.
\end{aligned}$$

Taking analogous arguments as the above inequality, we can obtain

$$1 + \sum_{p=\alpha+1}^{\alpha+\beta-1} (\lambda_{\alpha+\beta} - \lambda_p) a_{\alpha p}^2 \leq \frac{\lambda_{\alpha+\beta} - \lambda_1}{\lambda_{\alpha+1} - \lambda_1}.$$

From (2.25), the above inequalities and the lemma 2.2, we have

$$\begin{aligned}
B &\geq \sum_{\beta=1}^n \sum_{\alpha=1}^{n+1-\beta} \frac{(\lambda_{\alpha+\beta+1} - \lambda_{\alpha+\beta})(\lambda_{\alpha+1} - \lambda_1)^2}{(\lambda_{\alpha+\beta+1} - \lambda_1)(\lambda_{\alpha+\beta} - \lambda_1)} \left[1 - \sum_{p=\alpha+1}^{\alpha+\beta} (\lambda_p - \lambda_1) a_{\alpha p}^2 \right] \\
&\geq \sum_{\beta=1}^n \sum_{\alpha=1}^{n+1-\beta} \frac{(\lambda_{\alpha+\beta+1} - \lambda_{\alpha+\beta})(\lambda_2 - \lambda_1)^2}{(\lambda_{\alpha+\beta+1} - \lambda_1)(\lambda_{\alpha+\beta} - \lambda_1)} \left[1 - \sum_{p=\alpha+1}^{\alpha+\beta} (\lambda_p - \lambda_1) a_{\alpha p}^2 \right] \\
&= \sum_{\beta=2}^{n+1} \frac{(\lambda_{\beta+1} - \lambda_{\beta})(\lambda_2 - \lambda_1)^2}{(\lambda_{\beta+1} - \lambda_1)(\lambda_{\beta} - \lambda_1)} \sum_{\alpha=1}^{\beta-1} \left[1 - \sum_{p=\alpha+1}^{\beta} (\lambda_p - \lambda_1) a_{\alpha p}^2 \right].
\end{aligned}$$

On the other hand, for $\frac{\lambda_2}{\lambda_1} \geq 2 - \frac{\lambda_1}{\lambda_i}$, we have

$$\frac{\lambda_i}{\lambda_1} \geq \frac{\lambda_i - \lambda_1}{\lambda_2 - \lambda_1}.$$

Hence, from the lemma 2.3, we deduce

$$\begin{aligned}
C &\leq \frac{(\lambda_i - \lambda_1)}{\left(\frac{\lambda_i}{\lambda_1}\right)^2} \sum_{j=2}^{i-1} \frac{\lambda_i - \lambda_j}{\lambda_j - \lambda_1} \left[1 - (\lambda_j - \lambda_1) \sum_{\alpha=1}^{j-1} a_{\alpha j}^2 \right] \\
&= \frac{(\lambda_i - \lambda_1)}{\left(\frac{\lambda_i}{\lambda_1}\right)^2} \sum_{j=2}^{i-1} \frac{(\lambda_i - \lambda_1)(\lambda_{j+1} - \lambda_j)}{(\lambda_{j+1} - \lambda_1)(\lambda_j - \lambda_1)} \sum_{\alpha=1}^{j-1} \left[1 - \sum_{p=\alpha+1}^j (\lambda_p - \lambda_1) a_{\alpha p}^2 \right] \\
&\leq \sum_{j=2}^{i-1} \frac{(\lambda_2 - \lambda_1)^2 (\lambda_{j+1} - \lambda_j)}{(\lambda_{j+1} - \lambda_1)(\lambda_j - \lambda_1)} \sum_{\alpha=1}^{j-1} \left[1 - \sum_{p=\alpha+1}^j (\lambda_p - \lambda_1) a_{\alpha p}^2 \right] \\
&\leq \sum_{j=2}^{n+1} \frac{(\lambda_2 - \lambda_1)^2 (\lambda_{j+1} - \lambda_j)}{(\lambda_{j+1} - \lambda_1)(\lambda_j - \lambda_1)} \sum_{\alpha=1}^{j-1} \left[1 - \sum_{p=\alpha+1}^j (\lambda_p - \lambda_1) a_{\alpha p}^2 \right] \leq B.
\end{aligned}$$

Finally, we obtain, from (2.24) and the above inequalities,

$$\sum_{\alpha=1}^n (\lambda_{\alpha+1} - \lambda_1) \leq 3\lambda_1 + \frac{\lambda_1^2}{\lambda_i}.$$

This completes the proof of the theorem 1.1. \square

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